Int. J. Solids Structures Vol. 14, pp. 475–497
© Pergamon Press Ltd., 1978. Printed in Great Britain

0020-7683/78/0601-475/$02.00/0

A NEW APPROACH TO THE THEORY OF PLASTIC DEFORMATION

A. TATE
Procurement Executive, Ministry of Defence, Royal Armament Research and Development Establishment,
Fort Halstead, Sevenoaks, Kent TN14 7BP, England

(*Received* 21 *June* 1977; *in revised form* 21 *November* 1977; *received for publication* 3 *January* 1978)

Abstract—A simple, energy based, theory of plastic deformation is proposed and the extent to which it can model the known characteristics of the response of real materials is briefly explored. It is shown that a yield surface arises from the theory in a natural way and, in a particular case, takes the standard von Mises form. Post yield behaviour including the Bauschinger effect can also be incorporated within the framework of the theory. After first presenting the physical motivation the theory is developed for the case of infinitesimal one dimensional strain. Next the formalism is extended to an infinitesimal tensor strain field and finally to a finite tensor strain field.

1. INTRODUCTION

The earlier theories of plasticity, which are reviewed in Hill's classical treatise[1], are based on the following assumptions.

(a) There exists a scalar "yield function" which, when equated to zero, represents a closed surface in stress space. For stresses within the initial yield surface the deformation is completely elastic while for stresses outside the yield surface some plastic deformation occurs and unloading recovery is not complete.

(b) It is also usually assumed that the yielding is, to a first approximation unaffected by a moderate hydrostatic pressure or tension.

(c) The incremental components of the plastic strain tensor are proportional to the corresponding deviatoric stress components.

(d) The deformation for stresses outside the yield surface are governed by further "hardening rules" which again can usually be stated in terms of a hardening function.

A large literature has developed over the succeeding years in which the implications of the above assumptions have been examined in detail. Indeed it is difficult to find any modern treatment of macroscopic plastic deformation which is not set within the framework of the above assumptions, a notable exception being the work of Fitzgerald[2]. For many practical problems the classical theory has proved to be of great use however, as Wempner and Aberson[3] state, "*... the classical theory of plasticity has existed for decades in a generally acceptable form yet few formal solutions have been obtained for numerous problems of technological importance*". In three important respects classical plasticity theory has not proved entirely adequate. Firstly the incremental nature of assumption (c) has overstressed the rate characteristics of plastic flow and clouded the connexion with finite deformation theory. Secondly the theory as set out cannot immediately be integrated into the conceptual framework of modern physics in that most branches of modern theoretical physics deal primarily with the invariance, symmetry and stability properties connected with the energy and momentum of a system. It has thus proved difficult to correlate the microstructural features with the macroscopic deformation response. Lastly, it is not well suited to explaining recent experimental results on multiple deformation modes as discussed by Bell[4, 5]. It would thus appear allowable to consider if there exist other theoretical frameworks which may lead to more progress in these three respects.

The experimental basis for assumption (c) is not very clear and indeed experiments by Phillips and Gray[6] suggest it may not be strictly true. However, once the incremental concept is accepted it is a natural progression, though as Hill points out not a necessary one, to associate it with rate of change and indeed most dislocation theories are related to macroscopic response through a rate equation. This is unfortunate as the principal experimental plastic stress–strain response with which we will be concerned is essentially a quasi-static phenomenon. As Gilman[7] states dislocation acceleration times are very small (10^{-12}s) and, except for low stresses, the speed of propagation is a considerable fraction of the elastic wave speed. The formation time of a new state is thus far smaller than the loading rate except possibly during very high rate loading such as shock waves and, as the experimental work assembled by Bell[4] shows, for a wide range of strain rates the stress–strain relation is essentially independent of the strain rate. As there thus appears to be no compelling reason to couch our ideas in incremental terms it is here proposed to examine anew the possibilities of constructing a total deformation theory in the light of recent advances in continuum mechanics. The reason why this is not necessarily a futile quest was stated by Budiansky[8]. He recognized that the limitations of the deformation theories then current were that they ignored the possibility that the law governing the plastic response might exhibit independence of the stress history for a limited range of loading regimes and concluded that, for a total loading path, a deformation theory is a respectable competitor to other theories and should not be subjected to *a priori* rejection on physical grounds.

Within the last 10–20 yr there has been an upsurge of interest in formulating and developing in a rigorous way a general theory of the finite deformation of materials. Truesdell and Toupin[9] and Truesdell and Noll[10] reviewed the earlier work and set the pattern for most of the more recent developments in this field which has come to be known as continuum mechanics. The response of particular materials is given by their constitutive equations which relate the strain and various strain rates developed by the material to the locally applied stress. The theory sets out general principles which the constitutive equations must satisfy if they are to be physically acceptable. Within the general framework there is still a great deal of freedom in fixing the form of the constitutive equations and to make further progress these equations are classified according to their algebraic or differential structure. Whether such a classification is useful or not will depend on whether the mathematical structure can be directly related to chemical and microstructural characteristics of the material. Several particular forms of the continuum theory have been generated aimed at producing a plasticity theory but they are usually framed in general conformity with the classical plasticity assumptions as stated above (Refs.[11–15], to mention but a sample of the more recent work. In Section 2 we will introduce three postulates which may in certain instances, form a useful supplement to the general principles in determining practical constitutive equations and in relating them to the materials microstructural constitution.

It is now almost fifty years since it was first proposed that dislocation movement was the primary source of plastic deformation and within that time experimental techniques have advanced to the extent of actually showing a dislocation in an atomic lattice! However, as Nabarro[16] points out in his comprehensive review of the theory of dislocations the theory still stops short of a clear description of yielding in general. Although many effects can be qualitatively explained by dislocation theory the complex interactions between many different dislocations moving along differently oriented slip planes in the presence of other microstructural features such as grain boundaries, interstitial atoms, inclusions, etc. makes any detailed calculation very complex and often intractable. The individual dislocations are often known to form entanglements and forests, etc. so producing new features having a permanence which may make it profitable to regard them as the governing sources for macroscopically observable plastic deformation instead of the more basic dislocation. In the same way the dislocation is merely a relatively permanent feature of the even more basic atomic structure. In section 2 we will develop this idea and show how it indicates a set of postulates to supplement the continuum theory and make it more amenable to incorporation within this type of structured response. The remainder of the paper is devoted to exploring the resultant plastic theory. The overall response of a metal to a wide range of stress levels, loading rates and temperatures has experimentally been observed to be very complex. In this paper we will restrict our attention to ranges of the above variables for which dynamic and thermal effects can effectively be ignored as I feel these cloud the simplicity of the approach that is being proposed. That the range of these variables is not negligible is witnessed by the considerable wealth of experimental data presented by Bell[4,5]. The sum of observations presented in these books provides a severe critique for any theory and many aspects, such as the Savart–Masson effect cannot be incorporated in the proposed theory in an obvious way at present. However, Bell does find that both the elastic and plastic response consists of a multiplicity of deformation modes rather than just one and the theory proposed here is capable of taking account of this observation.

2. MICROSTRUCTURAL MOTIVATION FOR CONTINUUM THEORY

The most extensively studied relation between the microstructural interactions and the macroscopic response of a material has been for the case of elastic deformation. In this case it is assumed that a macroscopically homogeneous deformation can also be regarded as microscopically homogeneous, so that the atoms within the region all separate or close to the same extent. The process is reversible so that the mapping between initial and current positions of a particle is 1–1 and, in a simply connected region of space, the usual compatibility relations apply. The undeformed state is taken to be one in which the potential energy of the system is a minimum and thus depends at least quadratically on the strain measure.

When the shearing component of the strain becomes sufficiently large it becomes energetically favourable for microscopic inhomogeneities to be created within the discrete atomic lattice and for their regions of influence to spread in the form of dislocations. The macroscopic deformation associated with dislocation movement is termed plastic deformation. On an atomistic scale the deformation is not 1–1 and is not reversible. However, when a large number of dislocations are activated it is then possible to define a meaningful statistical average of plastic displacement, which contributes to the total macroscopic displacement and is in this statistical sense 1–1 but cannot be directly associated with atomic movements. For pure single crystals, particularly of hexagonal metals, the dislocations are constrained to lie in parallel glide planes and once they start moving they pass through the crystal with very little work hardening giving rise to Stage I or "laminar" plastic deformation. More generally dislocations travelling on different glide planes interact with one another leading to mutual restriction of movement and the formation of entanglements or dislocation forests or small regions of slightly misaligned crystal which are called, in the terminology of X-ray crystallography, mosaic structures. This type of deformation has been graphically termed "turbulent" plastic deformation. A question of current interest is to what extent do these larger structures rather than dislocations *per se* govern the materials macroscopic plastic response. If such structures do prove to be of prime importance then by considering a small region of space, which nevertheless contains a large number of such structures, one may associate, in a statistical sense, another plastic displacement component giving the contribution of this structural movement to the overall macroscopic response.

The above discussion has been developed with single crystals primarily in mind. Most metals and alloys in common use are however polycrystalline and consist of a conglomerate of small grains which may typically range from a few microns to above 1 mm in diameter. Again it may be possible to define a meaningful average grain displacement by considering the average response within a spherical region centred on the point of interest and large enough to contain a sufficiently large number of grains that the deviation from the mean is small.

The macroscopic deformation of a material may thus be regarded as consisting of the superposition of several different components each acting on a different scale. Just such a hierarchy of structures, created by a force driving the system far beyond the conditions for thermodynamic equilibrium, has been proposed by Glansdorf and Prigogine[17], though in a rather different context. Each scale of structure will depend on the smaller scale structural units within it but if it is to be a recognisable unit with a significant lifetime then it must not be strongly coupled to the other scales in the sense that its energy of formation and interaction must depend primarily on the deformation appropriate to its scale alone. The atomic scale, which in the context of this paper is associated with elastic displacement, has no energy of formation or more correctly it is convenient to measure all other energies relative to this energy. For other structures their energy of formation may be expected to be proportional to their number which is also proportional to their contribution to the plastic strain. The interaction energy between structures may be expected to be proportional to the number of close encounters between such structures. This in turn is proportional to the square of the number of these structures and thus is proportional to the square of the appropriate component of plastic strain. The total energy will be the sum of the energies of each component and the degree to which each component structure will be brought into play may be expected to be such that the total energy is a minimum. With the above physical model as motivation we put forward three postulates on which a theory of plasticity may be based.

First postulate. That the total strain experienced by the body can be regarded as being composed of the superposition of an elastic component and one or more plastic components.

Second postulate. The energy density associated with each component depends only on the strain associated with that component and the total energy density is the sum of the energy densities of all the components.

That the plastic and elastic energy contributions are almost independent is implied by experiments on the unloading and reloading of metals. On unloading the metal behaves elastically and on reloading immediately it follows almost the same stress–strain curve and continues along the plastic portion of the original stress–strain curve as though there had been no interruption. This behaviour occurs for varying amounts of plastic strain indicating that the elastic behaviour is not greatly affected by the varying plasticity induced. Provided the stress involved during loading does not exceed twice the initial yield stress then, on unloading to zero stress, the work done, as measured by the area under the stress–strain curve, is a measure of the plastic work involved in producing the residual amount of plastic strain. It should be noted that, although it is assumed that the plastic energy depends only on the plastic strain, it is not assumed that the plastic energy is a single valued function of the plastic strain. Indeed, when unloading is considered, it will generally be assumed that it is not single valued. A possible form for the energy during unloading is presented in the next section. In the primitive form to be presented in this paper no attempt is made to describe such rate effects as creep or the time dependent raising of the elastic yield limit.

Third postulate. The component strains are such as to make the total energy a minimum.

For this essentially mechanical theory, if the atoms can be redistributed, subject to a given total strain constraint, in such a way that the energy of the system is lowered, then either there exist internal forces driving the system towards this state or else the system is in a state of unstable equilibrium. Small thermal fluctuations militate against any prolonged stay in a position of unstable equilibrium and dissipative mechanisms will tend to ensure the system moves steadily towards the stable state.

At present the above principles can only be postulated and cannot be rigorously related to any more basic principles. Indeed it is realized that they can only be approximations to a more comprehensive theory which would have to include both kinetic and thermodynamic effects. However, a new approach to the relevant basic statistical mechanics incorporating a hierachy of structures related to forced instabilities as proposed by Glansdorf and Prigogine[17] would be required before a satisfactory basis for such a theory could be established.

3. ONE DIMENSIONAL INFINITESIMAL STRAIN THEORY OF PLASTICITY

Although, from a fundamental point of view, the above title seems to be inconsistent yet we can display all the ideas most simply in this context without being encumbered with tensor algebra and indeed much of the experimental and engineering work has been done on a one dimensional basis. Let ϵ be the total strain, ϵ^e be the elastic component and ϵ^p be the plastic component and set

$$\epsilon = \epsilon^e + \epsilon^p. \tag{3.1}$$

Let W^e be the energy density due to elastic deformation and W^p be the energy density due to plastic deformation. Then in accord with the second postulate we can write

$$W = W^e(\epsilon^e) + W^p(\epsilon^p). \tag{3.2}$$

Considerable simplification occurs now if we consider only linear elastic strain and plastic energy density only up to second order in the plastic strain.
Thus we take

$$W^e = \frac{1}{2}E\epsilon^{e^2} \tag{3.3}$$

where E is Young's modulus

and

$$W^p = A_1|\epsilon^p| + \frac{1}{2}A_2\epsilon^{p^2}. \tag{3.4}$$

The total strain energy density W is then given by

$$W = \frac{1}{2}E\epsilon^{e^2} + A_1|\epsilon^p| + \frac{1}{2}A_2\epsilon^{p^2} \tag{3.5}$$

or using eqn (3.1)

$$W = \frac{1}{2}E(\epsilon - \epsilon^p)^2 + A_1|\epsilon^p| + \frac{1}{2}A_2\epsilon^{p^2}. \tag{3.6}$$

We first consider the case of increasing total strain only and this implies increasing plastic strain, thus $\epsilon^p \geqslant 0$ and eqn (3.6) can be written as

$$W = \frac{1}{2}E(\epsilon - \epsilon^p)^2 + A_1\epsilon^p + \frac{1}{2}A_2\epsilon^{p^2}. \tag{3.7}$$

In accord with the third postulate we seek to find the minimum value of W for a given total strain ϵ and setting $\partial W/\partial\epsilon^p$ to zero gives

$$\epsilon^p = (E\epsilon - A_1)/(E + A_2) \quad \epsilon^p > 0. \tag{3.8}$$

For values of ϵ less than (A_1/E), the minimum value of W occurs when ϵ^p is zero and thus for strains below this value only elastic deformation occurs. The stress is given by the usual expression

$$T = \frac{\mathrm{d}W}{\mathrm{d}\epsilon}. \tag{3.9}$$

In the elastic region this gives $T = E\epsilon$ and at the strain above which plastic deformation occurs, and is therefore the yield point, $T = A_1$. Thus we may associate A_1 with the yield stress. For strains beyond the yield strain substitution of (3.8) into (3.7) gives the energy density as

$$W = (A_2E\epsilon^2 + 2A_1E\epsilon - A_1^2)/2(E + A_2). \tag{3.10}$$

The associated stress in the plastic region is then given by

$$T = \frac{\mathrm{d}W}{\mathrm{d}\epsilon} = E(A_1 + A_2\epsilon)/(E + A_2) \tag{3.11}$$

giving

$$\frac{\mathrm{d}T}{\mathrm{d}\epsilon} = \frac{EA_2}{E + A_2} = 1/\left[\frac{1}{E} + \frac{1}{A_2}\right]. \tag{3.12}$$

Thus A_2 can be associated with a linear work hardening parameter. Substituting (3.8) into (3.1) gives

$$\epsilon^e = (A_1 + A_2\epsilon)/(E + A_2) \tag{3.13}$$

and using this expression in eqn (3.11) gives the simple relation

$$T = E\epsilon^e. \tag{3.14}$$

Thus there is no discontinuity in the stress as the total strain is increased beyond the yield point. That stress is always simply related to the elastic component is shown by the following argument. If ϵ^e is replaced in (3.2) by using (3.1) then W can be written in the form $W(\epsilon, \epsilon^p)$ and the third postulate then takes the form

$$\left(\frac{\partial W}{\partial \epsilon^p}\right)_\epsilon = 0 \tag{3.15}$$

where the subscript ϵ is used in the thermodynamic sense to denote what is being kept constant.

This defines ϵ^p as a function of ϵ and W takes the form $W(\epsilon, \epsilon^p(\epsilon))$. The stress is then given by

$$T = \frac{\mathrm{d}W}{\mathrm{d}\epsilon} = \left(\frac{\partial W}{\partial \epsilon}\right)_{\epsilon^p} + \left(\frac{\partial W}{\partial \epsilon^p}\right)_\epsilon \frac{\partial \epsilon^p}{\partial \epsilon}. \tag{3.16}$$

But using (3.15) and noting from (3.1) that $\partial\epsilon = \partial\epsilon^e$ for constant ϵ^p

$$T = \frac{\partial W}{\partial \epsilon^e}. \tag{3.17}$$

The using (3.2) gives finally

$$T = \frac{\partial W^e}{\partial \epsilon^e}. \tag{3.18}$$

The type of loading stress–strain curve which can be described by the theory so far is as shown in Fig. 1 and is a reasonable first approximation to the stress–strain curve of many metals.

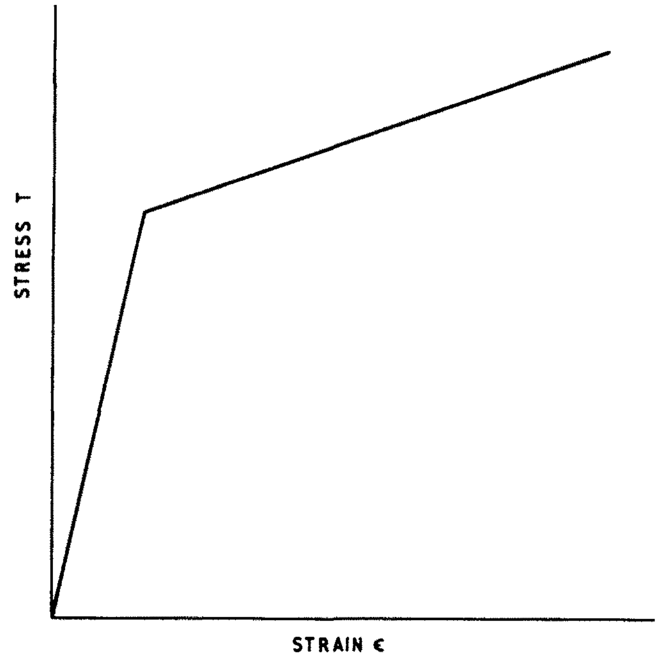

Fig. 1. Loading stress–strain curve for one activated plastic deformation structure. (Vertical axis; "Stress T"; Horizontal axis; "Strain ϵ").

However, it cannot represent, even approximately cases such as steel where a yield drop occurs. To get a better representation of the response of such materials let us suppose that plastic deformation mechanisms depending on two different structures are brought into play. Using the first two postulates of Section 1 we can write the total strain and the total strain energy density in the forms

$$\epsilon = \epsilon^e + \epsilon_1{}^p + \epsilon_2{}^p \tag{3.19}$$

and

$$W = W^e(\epsilon^e) + W_1{}^p(\epsilon_1{}^p) + W_2{}^p(\epsilon_2{}^p). \tag{3.20}$$

Using (3.19), (3.20) can be written in the form $W = W(\epsilon, \epsilon_1{}^p, \epsilon_2{}^p)$ and the third postulate then takes the form

$$\left(\frac{\partial W}{\partial \epsilon_1{}^p}\right)_{\epsilon,\epsilon_2{}^p} = 0; \quad \left(\frac{\partial W}{\partial \epsilon_2{}^p}\right)_{\epsilon,\epsilon_1{}^p} = 0 \tag{3.21}$$

and the stress is given by

$$T = \frac{\mathrm{d}W(\epsilon, \epsilon_1{}^p(\epsilon), \epsilon_2{}^p(\epsilon))}{\mathrm{d}\epsilon} = \left(\frac{\partial W}{\partial \epsilon}\right)_{\epsilon_1{}^p,\epsilon_2{}^p} + \left(\frac{\partial W}{\partial \epsilon_1{}^p}\right)_{\epsilon,\epsilon_2{}^p} \frac{\partial \epsilon_1{}^p}{\partial \epsilon} + \left(\frac{\partial W}{\partial \epsilon_2{}^p}\right)_{\epsilon,\epsilon_1{}^p} \frac{\partial \epsilon_2{}^p}{\partial \epsilon} \tag{3.22}$$

or using eqns (3.19)–(3.21)

$$T = \frac{\partial W^e}{\partial \epsilon^e}. \tag{3.23}$$

Again it is convenient to simplify the presentation by considering that the energy densities depend on at most the second order strains and that the strains are always increasing.
Thus

$$W = \frac{1}{2} E\ \epsilon^{e^2} + A_1(1)\epsilon_1{}^p + \frac{1}{2}A_2(1)\epsilon_1{}^{p^2} + A_1(2)\epsilon_2{}^p + \frac{1}{2}A_2(2)\epsilon_2{}^{p^2}. \tag{3.24}$$

To find the minimum value of W subject to (3.19) we use Lagrange's method of undetermined multipliers and set the coefficient multiplying each differential coefficient to zero giving

$$E\epsilon^e = \zeta = T \quad \text{or} \quad \epsilon^e = \zeta/E \tag{3.25a}$$

$$A_1(1) + A_2(1)\epsilon_1{}^p = \zeta = T \quad \text{or} \quad \epsilon_1{}^p = (\zeta - A_1(1))/A_2(1) \tag{3.25b}$$

$$A_1(2) + A_2(2)\epsilon_2{}^p = \zeta = T \quad \text{or} \quad \epsilon_2{}^p = (\zeta - A_1(2))/A_2(2) \tag{3.25c}$$

where

$$\zeta = \frac{\epsilon + (A_1(1)/A_2(1)) + (A_1(2)/A_2(2))}{(1/E) + (1/A_2(1)) + (1/A_2(2))} = T \tag{3.26}$$

and use has been made of eqn (3.23). Thus ζ which equals T varies monotonically with ϵ and, if $A_1(2) > A_1(1)$, only the first plastic mode is excited when $A_1(1) < T < A_1(2)$; both plastic modes are excited if $T > A_1(2)$; and no plastic modes are excited when $T < A_1(1)$. The stress gradient is easily seen to have the form

$$\frac{\mathrm{d}T}{\mathrm{d}\epsilon} = 1\Big/\left[\frac{1}{E} + \frac{1}{A_2(1)} + \frac{1}{A_2(2)}\right], \qquad T > A_1(2) \tag{3.27a}$$

$$\frac{dT}{d\epsilon} = 1\Big/\left[\frac{1}{E} + \frac{1}{A_2(1)}\right], \qquad A_1(1) < T < A_1(2). \tag{3.27b}$$

The type of stress–strain curve generated by the above theory is as shown in Fig. 2(a).

As the second plastic mode depends on the first mode it may happen that a certain amount of first mode deformation must occur before the second mode can even be formed. If this is so a yield drop situation could arise in the following way. First let us note that, when only one plastic mode is excited, the condition for W to be a minimum, and therefore in a stable condition, is that

$$\frac{d^2W}{d{\epsilon_1^p}^2} = E + A_2(1) > 0. \tag{3.28}$$

Thus the stress gradient, which has the form

$$\frac{dT}{d\epsilon} = \frac{EA_2(1)}{E + A_2(1)} \tag{3.29}$$

can be negative even up to the limit $(dT/d\epsilon) \to -\infty$. By taking $A_2(1)$ to be negative, $A_1(2)$ to be less than $A_1(1)$ and $\epsilon_2^p = 0$ if $\epsilon_1^p < [A_1(2) - A_1(1)]/A_2(1)$ we have the possibility of describing a yield drop as shown in Fig. 2(b). The upper and lower yield stress values are given by $A_1(1)$ and $A_1(2)$ respectively while if the gradient of the stress–strain line beyond the first gradient

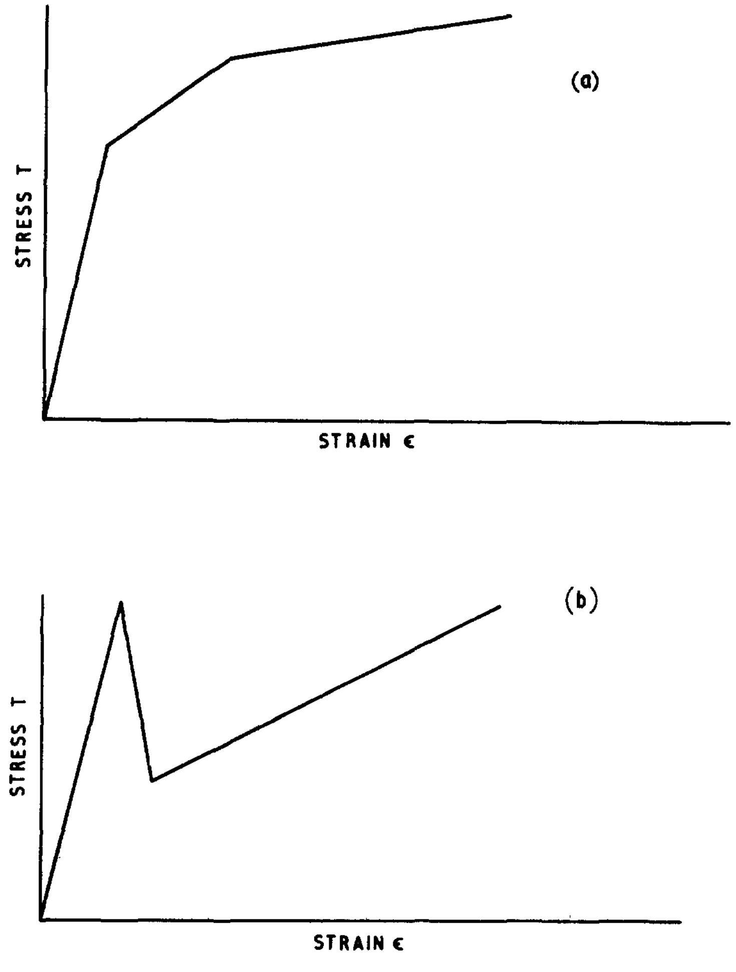

Fig. 2. Two types of loading stress–strain curve which can be generated if two plastic deformation structures are activated. (a) and (b). (Vertical axis: "Stress T"; Horizontal axis: "Strain ε").

discontinuity is m_1 and beyond the second discontinuity is m_2 then

$$\frac{1}{A_2(1)}=\frac{1}{m_1}-\frac{1}{E} \quad \text{or} \quad A_2(1)=\frac{m_1E}{E-m_1} \tag{3.30}$$

$$\frac{1}{A_2(2)}=\frac{1}{m_2}-\frac{1}{m_1} \quad \text{or} \quad A_2(2)=\frac{m_1m_2}{m_1-m_2}. \tag{3.31}$$

If N components of plastic deformation are successively excited then in the second order strain approximation the total energy W is given by

$$W=\frac{1}{2}E\epsilon^{e2}+\sum_{r=1}^{N}\left[A_1(r)\epsilon_r^p+\frac{1}{2}A_2(r)\epsilon_r^{p^2}\right]. \tag{3.32}$$

The stress–strain curve will thus consist of $N+1$ linear sections.

The stress at the rth join between sections will be at $T=A_1(r)$ and the strain is given by

$$\epsilon=\frac{A_1(r)}{E}+\sum_{j=1}^{r}\frac{A_1(r)-A_1(j)}{A_2(j)} \tag{3.33}$$

and if the gradient of the line after the rth join is denoted by m_r

$$\frac{1}{m_r}=\frac{1}{E}+\sum_{j=1}^{r}\frac{1}{A_2(j)} \tag{3.34a}$$

then

$$\frac{1}{A_2(r)}=\frac{1}{m_r}-\frac{1}{m_{r-1}}. \tag{3.34b}$$

If there is a continuous distribution of components the total energy W to the second order strain approximation is given by

$$W=\frac{1}{2}E\epsilon^{e^2}+\int_0^R\left[A_1(s)\bar{\epsilon}^p(s)+\frac{1}{2}A_2(s)\bar{\epsilon}^{p^2}(s)\right]\mathrm{d}s \tag{3.35}$$

while the total strain is given by

$$\epsilon=\epsilon^e+\int_0^R\bar{\epsilon}^p(s)\,\mathrm{d}s \tag{3.36}$$

where $\bar{\epsilon}^p(s)$ is the density of plastic strain per unit range of activated component states. From the above the stress–strain relation is generated in the parametric form

$$T=A_1(\eta) \tag{3.37a}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}T}=\frac{1}{E}+\int_0^\eta\frac{\mathrm{d}s}{A_2(s)}. \tag{3.37b}$$

$A_1(\eta)$ is associated with the energy of formation of the ηth state and if we choose to label our states in terms proportional to the energy then

$$A_1(\eta)=K\eta \qquad (K \text{ is a constant}). \tag{3.38}$$

If $A_2(\eta)$ is then taken to be of the form

$$A_2(\eta)=B\exp(-\beta\eta) \qquad (B \text{ and } \beta \text{ are constants}). \tag{3.39}$$

Then

$$T = K\eta \qquad \eta \leqslant R \tag{3.40a}$$

$$T = T_a \qquad \eta = R \tag{3.40b}$$

$$\frac{d\epsilon}{dT} = \frac{1}{E} + \frac{1}{B\beta}\exp(\beta\eta) - \frac{1}{B\beta} \qquad T \leqslant T_a \tag{3.40c}$$

and when

$$B\beta = E, \frac{d\epsilon}{dT} = \frac{1}{E}\exp(\beta\eta) = \frac{1}{E}\exp\left(\frac{\beta T}{K}\right) = \frac{1}{E}\exp\left(\frac{T}{T_a}\right) \tag{3.41}$$

or

$$\frac{dT}{d\epsilon} = \frac{E}{[1 + (\epsilon/\epsilon_0)]} \qquad T \leqslant T_a \tag{3.42}$$

where we define

$$T_0 = K/\beta \quad \text{and} \quad \epsilon_0 = T_0/E.$$

As in previous sections for $\epsilon > \epsilon_a$ the stress–strain relation is linear. The above relations correspond to the plastic deformation relation examined by Billington and Tate[18] which has a stress–strain curve as shown in Fig. 3. Of particular interest in relation to the present theory is the experimental work of Bell which he has presented in detail and in their historical setting in two monographs[4, 5]. In these works Bell shows that for a great many different crystalline solids there exists a uniaxial stress–strain relation which, when expressed in nominal form and

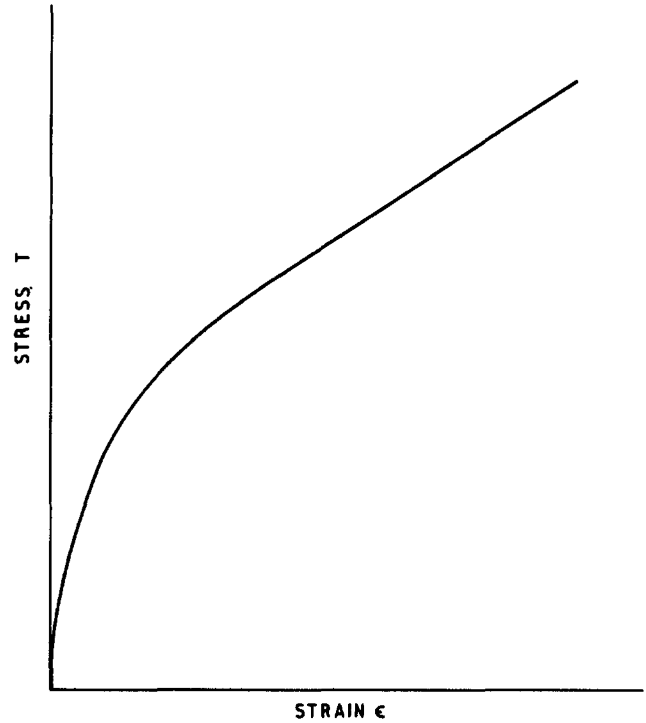

Fig. 3. Type of stress–strain relation found by Billington and Tate[18]. (Vertical axis: "Stress T"; Horizontal axis: "Strain ϵ").

referred to the undeformed state, consists of several parabolic segments of the form

$$T = (2/3)^{r/2}\mu(0)B_0(1-\theta/\theta_m)(\epsilon-\epsilon_b)^{1/2} \tag{3.43}$$

where T is the nominal uniaxial stress, ϵ is the nominal uniaxial strain, ϵ_b characterises a particular parabola, B_0 is a dimensionless universal constant, θ is the absolute temperature of the specimen, θ_m the absolute melting temperature of the specimen material and $\mu(0)$ is the zero point isotropic elastic shear modulus.

First we must extend slightly the presentation given above to the case of several sets of continuous distributions of plastic components. That is, we consider the total energy to be of the form

$$W = \frac{1}{2}E\epsilon^{e^2} + \sum_{j=1}^{N}\int_{R_0{}^j}^{R_1{}^j}\left[A_1{}^j(s)\bar{\epsilon}^p(s) + \frac{1}{2}A_2{}^j(s)\bar{\epsilon}^{p^2}(s)\right]\mathrm{d}s \tag{3.44}$$

while the total strain is given by

$$\epsilon = \epsilon^e + \sum_{j=1}^{N}\int_{R_0{}^j}^{R_1{}^j}\bar{\epsilon}^p(s)\,\mathrm{d}s. \tag{3.45}$$

Within each "j" deformation mode the stress–strain relation is generated in the parametric form

$$T = A_1{}^j(\eta) \tag{3.46}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}T} = \frac{1}{E} + \int_{R_0{}^1}^{R_1{}^1}\frac{\mathrm{d}s}{A_2{}^1(s)} + \ldots + \int_{R_0{}^j}^{\eta}\frac{\mathrm{d}s}{A_2{}^j(s)}. \tag{3.47}$$

Now $A_1{}^j(\eta)$ is associated with the energy density of formation of the ηth state and if we choose to label our states in terms proportional to the energy density then

$$A_1{}^j(\eta) = K_j\eta \quad (K_j \text{ are constants}). \tag{3.48}$$

If the "j" segment corresponds to the parabolic deformation mode of eqn (3.43) then

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}T} = 2T\Big/\left(\frac{2}{3}\right)^{r/2}\mu(0)B_0(1-\theta/\theta_m). \tag{3.49}$$

Thus (3.47) can be written in the form

$$L_j\eta = \frac{1}{E} + \int_{R_0{}^1}^{R_1{}^1}\frac{\mathrm{d}s}{A_2{}^1(s)} + \ldots + \int_{R_0{}^j}^{\eta}\frac{\mathrm{d}s}{A_2{}^j(s)} \tag{3.50}$$

where

$$L_j = 2K_j\Big/\left(\frac{2}{3}\right)^{r/2}\mu(0)B_0(1-\theta/\theta_m). \tag{3.51}$$

Differentiating (3.50) w.r.t. "η" then gives

$$A_2{}^j(\eta) = \frac{1}{L_j} = \left(\frac{2}{3}\right)^{r/2}\mu(0)B_0\left(1-\frac{\theta}{\theta_m}\right)\Big/2K_j. \tag{3.52}$$

Thus within each "j" mode the second order interaction term would be simply a constant. If indeed we label our states in terms of the energy density $A_1{}^j(\eta) = \eta$ then $A_2{}^j$ can be written in the form

$$A_2{}^j(\eta) = \frac{1}{2}\left(\frac{2}{3}\right)^{r/2}\mu_0B_0\left(1-\frac{\theta}{\theta_m}\right). \tag{3.53}$$

Having considered a loading pulse let us now consider unloading. Let the plasticity induced during loading be denoted by $\epsilon^{p\uparrow}$ and that during unloading by $\epsilon^{p\downarrow}$. The total plastic strain is then given by

$$\epsilon^p = \epsilon^{p\uparrow}_{\max} - \epsilon^{p\downarrow} \quad \text{(Note } \epsilon^{p\downarrow} \text{ will be positive numerically).} \tag{3.54}$$

To make some estimate of the energy of deformation it would seem reasonable to expect that dislocations or other structures, after they have been formed, will become locked in entanglements and cannot easily be destroyed except through thermal agitation. If this is true it may be energetically more favourable to create new dislocations or other structures giving rise to plastic strain in the other direction. Thus, to second order in the plastic strain, it will be assumed that on unloading the plastic component of the energy $W^{p\downarrow}$ is given by

$$W^{p\downarrow} = A_1\epsilon^{p\uparrow}_{\max} + A_1\epsilon^{p\downarrow} + \frac{1}{2}A_2\epsilon^{p^2} \tag{3.55}$$

$$= 2A_1\epsilon^{p\uparrow}_{\max} - A_1\epsilon^p + \frac{1}{2}A_2\epsilon^{p^2}. \tag{3.56}$$

Thus the total energy on unloading is given by

$$W^{\downarrow} = W^e + W^{p\downarrow} \tag{3.57}$$

or

$$W^{\downarrow} = \frac{1}{2}E\epsilon^{e^2} + 2A_1\epsilon^{p\uparrow}_{\max} - A_1\epsilon^p + \frac{1}{2}A_2\epsilon^{p^2}. \tag{3.58}$$

Applying the third postulate gives the minimum energy when

$$\epsilon^p = (E\epsilon + A_1)/(E + A_2) \tag{3.59}$$

or

$$\epsilon^{p\downarrow} = [E(\epsilon_{\max} - \epsilon) - 2A_1]/(E + A_2). \tag{3.60}$$

Thus as long as

$$\epsilon > \epsilon_{\max} - 2(A_1/E). \tag{3.61}$$

$\epsilon^{p\downarrow}$ must be zero and the recovery will be entirely elastic.

It may be noted that the arguments leading to the stress relation $T = (\partial W^e/\partial\epsilon^e)$ of (3.18) remain essentially unaltered except that ϵ^p must be replaced by $\epsilon^{p\downarrow}$ and thus in all cases

$$T = \frac{\partial W^e}{\partial \epsilon^e} \tag{3.62}$$

and in the particular case being considered

$$T = E\epsilon^e \tag{3.63}$$

and using (3.59) this gives for $\epsilon < \epsilon_{\max} - (2A_1/E)$

$$T = E(\epsilon A_2 - A_1)/(E + A_2) \tag{3.64}$$

or

$$\frac{\mathrm{d}T}{\mathrm{d}\epsilon} = EA_2/(E + A_2) \tag{3.65}$$

which, comparing with (3.12), is parallel to the plastic part of the loading curve. A typical curve of loading and unloading described by the above theory is as shown in Fig. 4 and this form of the theory clearly gives rise to a Bauschinger effect. Just such a loading curve has been found experimentally by Lin and Ito[19] who also give extensive reference to other experimental papers showing similar loading curves. If the loading is reversed before $\epsilon^{p\downarrow}$ is activated the loading curve will retrace the unloading elastic portion of the curve until the former maximum stress is reached after which it will continue along the plastic stress–strain curve as though there had been no unloading interruption.

If "k" modes are excited before unloading occurs then, provided the higher k modes do not inhibit activation of the lower modes, we can proceed as above

$$\epsilon_r^p = \epsilon_{r\max}^{p\uparrow} - \epsilon_r^{p\downarrow} \qquad r = 1, 2 \ldots k \tag{3.66}$$

and

$$W^{p\downarrow} = \sum_{r=1}^{k}\left(A_1{}^r\epsilon_{r\max}^{p\uparrow} + A_1{}^r\epsilon_r^{p\downarrow} + \frac{1}{2}A_2\epsilon_r^{p^2}\right) \tag{3.67}$$

giving a total energy density on unloading

$$W^{\downarrow} = \frac{1}{2}E\epsilon^{e^2} + \sum_{r=1}^{k}\left(2A_1{}^r\epsilon_{r\max}^{p\uparrow} - A_1{}^r\epsilon_r^p + \frac{1}{2}A_2{}^r\epsilon_r^{p^2}\right). \tag{3.68}$$

Using Langranges method of undetermined multipliers again and the third postulate gives

$$E\epsilon^e = \zeta = T \tag{3.69}$$

$$-A_1{}^r + A_2{}^r\epsilon_r^p = \zeta = T \qquad r = 1 \ldots k. \tag{3.70}$$

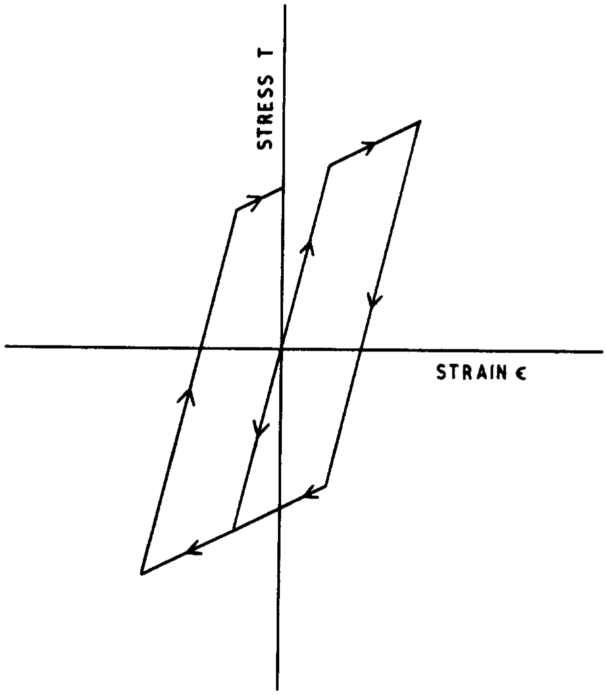

Fig. 4. Typical loading and unloading curve when one plastic deformation structure is activated. (Vertical axis: "Stress T"; Horizontal axis; "Strain ε").

Thus

$$\epsilon_r^p = (T + A_1{}^r)/A_2{}^r \tag{3.71}$$

and using (3.66)

$$\epsilon_r^{p\downarrow} = \epsilon_{r\max}^{p\uparrow} - \epsilon_r^p = \frac{T_{\max}^{\uparrow} - A_1{}^r}{A_2{}^r} - \frac{T + A_1{}^r}{A_2{}^r} \tag{3.72}$$

or

$$\epsilon_r^{p\downarrow} = (T_{\max}^{\uparrow} - T - 2A_1{}^r)/A_2{}^r. \tag{3.73}$$

Thus $\epsilon_r^{p\downarrow}$ is not activated until $T < T_{\max}^{\uparrow} - 2A_1{}^r$ and on unloading the stress–strain curve will mirror the original loading curve starting with the elastic part of the curve.

4. THREE DIMENSIONAL INFINITESIMAL STRAIN THEORY

The extension to three dimensions involves introducing the infinitesimal strain tensor $\boldsymbol{\epsilon}$ but otherwise follows in a straightforward manner the same development as the one dimensional theory. We start by considering only loading and assume the total strain can be expressed as the sum of an elastic and a plastic component

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p \tag{4.1}$$

and that the total energy W can also be expressed as a sum of the elastic and plastic energies

$$W = W^e(\boldsymbol{\epsilon}^e) + W^p(\boldsymbol{\epsilon}^p) = W(\boldsymbol{\epsilon}^e, \boldsymbol{\epsilon}^p) \tag{4.2}$$

or using (4.1)

$$W = W(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}^p). \tag{4.3}$$

Now it is convenient to express each tensor component as the sum of an isotropic and a deviatoric part

$$\boldsymbol{\epsilon} = \epsilon''\mathbf{1} + \boldsymbol{\epsilon}' \quad \text{(for each } \boldsymbol{\epsilon} \text{ tensor)} \tag{4.4}$$

thus

$$\epsilon'' = \epsilon^{e''} + \epsilon^{p''} \tag{4.5a}$$

and

$$\boldsymbol{\epsilon}' = \boldsymbol{\epsilon}^{e'} + \boldsymbol{\epsilon}^{p'}. \tag{4.5b}$$

In accord with the third postulate we find the minimum value of W for a given total strain $\boldsymbol{\epsilon}$ from (4.3) by setting

$$\left(\frac{\partial W(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}^p)}{\partial \boldsymbol{\epsilon}^p}\right)_{\boldsymbol{\epsilon}} = 0 \tag{4.6}$$

which then gives $\boldsymbol{\epsilon}^p$ as a function of $\boldsymbol{\epsilon}$.
The stress $\mathbf{T}$ is now given by

$$\mathbf{T} = \frac{\partial W(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}^p(\boldsymbol{\epsilon}))}{\partial \boldsymbol{\epsilon}} \tag{4.7}$$

or

$$\mathbf{T} = \left(\frac{\partial W}{\partial \boldsymbol{\epsilon}}\right)_{\boldsymbol{\epsilon}^p} + \left(\frac{\partial W}{\partial \boldsymbol{\epsilon}^p}\right)_{\boldsymbol{\epsilon}} \frac{\partial \boldsymbol{\epsilon}^p}{\partial \boldsymbol{\epsilon}}. \tag{4.8}$$

But using (4.6) and noting that $\partial \boldsymbol{\epsilon} = \partial \boldsymbol{\epsilon}^e$ if $\boldsymbol{\epsilon}^p$ is constant

$$\mathbf{T} = \frac{\partial W}{\partial \boldsymbol{\epsilon}^e} = \frac{\partial W^e}{\partial \boldsymbol{\epsilon}^e}. \tag{4.9}$$

Now resolving $\mathbf{T}$ into isotropic and deviatoric parts gives immediately

$$T'' = \frac{1}{3}\frac{\partial W^e}{\partial \epsilon^{e''}};\ \mathbf{T}' = \frac{\partial W^e}{\partial \boldsymbol{\epsilon}^{e'}}. \tag{4.10}$$

Restricting our discussion to isotropic materials, the linear elastic strain energy has the form

$$W^e = \frac{9}{2}\kappa \epsilon^{e''2} + G\boldsymbol{\epsilon}^{e'} : \boldsymbol{\epsilon}^{e'} \tag{4.11}$$

where κ is the bulk modulus and G the shear modulus or modulus of rigidity. It is usual to assume that plastic deformation occurs incompressibly in which case $\epsilon^{p''}$ is zero and for an isotropic material W^p takes the form

$$W^p = W^p(II'_p, III'_p) \tag{4.12}$$

where II'_p and III'_p are the second and third invariants of the deviatoric components of plastic strain. Now using (4.5) and (4.6) the total energy density takes the form

$$W = \frac{9}{2}\kappa \epsilon''^2 + G(\boldsymbol{\epsilon}' - \boldsymbol{\epsilon}^{p'}) : (\boldsymbol{\epsilon}' - \boldsymbol{\epsilon}^{p'}) + W^p(II'_p, III'_p). \tag{4.13}$$

From (4.5), (4.10) and (4.11) it follows that the isotropic or hydrostatic component of stress T'' is given by

$$T'' = 3\kappa\epsilon'' \quad (\epsilon^{p''} = 0) \tag{4.14}$$

and as, in seeking the minimum value of W from (4.13) keeping $\boldsymbol{\epsilon}$ constant, the term involving ϵ'' vanishes on differentation, it follows that the hydrostatic stress will not affect the yield stress. This result therefore is a rigorous consequence of the assumptions contained within the theory this far and does not have to be added as an extra assumption.

To find the minimum value of W set each partial derivative to zero

$$\left(\frac{\partial W}{\partial \boldsymbol{\epsilon}^{p'}}\right)_{\boldsymbol{\epsilon}} = -2G\boldsymbol{\epsilon} + 2G\boldsymbol{\epsilon}^{p'} + \frac{\partial W^p}{\partial II'_p}\frac{\partial II'_p}{\partial \boldsymbol{\epsilon}^{p'}} + \frac{\partial W^p}{\partial III'_p}\frac{\partial III'_p}{\partial \boldsymbol{\epsilon}^{p'}}. \tag{4.15}$$

Multiplying eqn (4.15) by $:\boldsymbol{\epsilon}^{p'}$, adding using the summation convention and using Eulers theorem on homogeneous functions (i.e. $\Sigma(\partial f/\partial x_j)x_j = nf$ if f is homogeneous of degree n in the x_j's) gives

$$-2G\boldsymbol{\epsilon}' : \boldsymbol{\epsilon}^{p'} + 4GII'_p + 2\frac{\partial W^p}{\partial II'_p} II'_p + 3\frac{\partial W^p}{\partial III'_p} III'_p = 0. \tag{4.16}$$

Now let us set

$$\boldsymbol{\epsilon}^{p'} = \lambda(II', III')\boldsymbol{\epsilon}' \tag{4.17}$$

where II' and III' are the second and third invariants of the total strain deviator. Thus

$$II'_p = \lambda^2 II' \quad \text{and} \quad III'_p = \lambda^3 III'. \tag{4.18}$$

Substituting in (4.16)

$$-4G\lambda II' + 4G\lambda^2 II' + 2\lambda^2 \left(\frac{\partial W^p}{\partial II'_p}\right)_{II'_p=\lambda^2 II'} II' + 3\lambda^3 \left(\frac{\partial W^p}{\partial III'_p}\right)_{III'_p=\lambda^3 III'} III' = 0. \tag{4.19}$$

Solving this equation specifies λ as a function of II' and III' and then using the values for ϵ' which make λ zero specifies the yield surface. In particular let us set

$$W^p = A_1 II_p'^{1/2} + 2A_2 II'_p. \tag{4.20}$$

Equation (4.19) then takes the form

$$-2G + 2G\lambda + \frac{1}{2}A_1(II')^{-1/2} + 2A_2\lambda = 0$$

or

$$\lambda = \{2G - [A_1/2(II')^{1/2}]\}/(2G + 2A_2). \tag{4.21}$$

The yield surface is thus given by

$$(II')^{1/2} = A_1/4G. \tag{4.22}$$

In the elastic region

$$\epsilon' = \epsilon^{e'} = \mathbf{T}'/2G \tag{4.23}$$

thus

$$II' = J_2/4G^2 \tag{4.24}$$

where J_2 is the second invariant of the stress tensor.

Substituting in (4.22) gives

$$J_2^{1/2} = A_1/2 \tag{4.25}$$

which is von Mises form of the yield criterion.

When the material is stressed beyond the yield surface the stress is still given by

$$\mathbf{T}' = \frac{\partial W^e}{\partial \epsilon^{e'}} = 2G\epsilon^{e'} = 2G(\epsilon' - \epsilon^{p'}). \tag{4.26}$$

Now using (4.17)

$$\mathbf{T}' = 2G(1-\lambda)\epsilon' \tag{4.27}$$

and from (4.21)

$$\mathbf{T}' = G[2A_2 + (A_1/2(II')^{1/2})]\epsilon'/(G + A_2). \tag{4.28}$$

By extending the plastic energy to have two or more dependent components, for example

$$W^p = A_1(1)II'^{1/2}_{p1} + 2A_2(1)II'_{p1} + A_1(2)II'^{1/2}_{p2} + 2A_2(2)II'_{p2} \tag{4.29}$$

one can produce stress–strain curves showing a yield drop and having two yielding surfaces.

The extension to a continuous distribution of states and to the case of unloading follows the same development as the one dimensional case discussed in Section 2, namely, in the case of a continuous distribution a density of states $\epsilon^p(s)$ must be introduced and when unloading occurs an unloading plastic strain $\epsilon^p(s)^\downarrow$ must then be introduced.

5. FINITE THREE DIMENSIONAL STRAIN THEORY

The plastic deformation can be very large, of the order 30% strain or even more and for such cases the theory developed above cannot be regarded as having more than qualitative significance. We therefore seek an extension of the above theory applicable to finite strain and falling within the framework of continuum mechanics as developed by Truesdell and his school but having its roots in the work of Reiner, Rivlin and Treloar and even back to the Cosserats. Let the initial or reference coordinates be denoted by X with components X^α and the current coordinates be x with components x^i. Let the total displacement be composed of a plastic displacement on top of which an elastic displacement is superimposed. Let the current plastic configuration be specified by coordinates X^p with components X^{pa}. Thus

$$x = x(X^p(X)) \quad \text{or} \quad x^i = x^i(X^{pa}(X^\alpha)). \tag{5.1}$$

The chain rule of differentiation then leads to the following relation between the deformation gradients

$$\frac{\partial x^i}{\partial x^\alpha} = \frac{\partial x^i}{\partial X^{pa}}\frac{\partial X^{pa}}{\partial X^\alpha} \quad \text{or} \quad F_\alpha{}^i = F_a{}^{ei} F_\alpha{}^{pa} \quad \text{or} \quad \mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{5.2}$$

where $\mathbf{F}$ is the total deformation gradient and $\mathbf{F}^e$ and $\mathbf{F}^p$ are the elastic and plastic components. Using the second postulate the total energy density can be written in the form

$$W = W^e + W^p \tag{5.3}$$

where W^e depends only on the elastic strain components and W^p depends only on the plastic strain components. Care must be taken in choosing the form which W^e takes however, because although it is assumed to be independent of the plastic strain it will be shown that it may have to depend on the plastic rotation, a point we will return to shortly.

Using the polar decomposition theorem the deformation gradients can be written in the following form:

$$\mathbf{F} = \mathbf{VR} = \mathbf{RU}; \qquad F_\alpha{}^i = V_j{}^i R_\alpha{}^j = R_\beta{}^i U_\alpha{}^\beta \tag{5.4}$$

$$\mathbf{F}^p = \mathbf{V}^p\mathbf{R}^p = \mathbf{R}^p\mathbf{U}^p; \qquad F_\alpha{}^{pa} = V_p{}^{pa} R_\alpha{}^{pb} = R_\beta{}^{pa} U_\alpha{}^{p\beta} \tag{5.5}$$

$$\mathbf{F}^e = \mathbf{V}^e\mathbf{R}^e = \mathbf{R}^e\mathbf{U}^e; \qquad F_a{}^{ei} = V_j{}^{ei} R_a{}^{ej} = R_b{}^{ei} U_a{}^{eb}. \tag{5.6}$$

Using the above

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p = \mathbf{V}^e\mathbf{R}^e\mathbf{R}^p\mathbf{U}^p = \mathbf{VR} = \mathbf{RU}. \tag{5.7}$$

Defining

$$\bar{\mathbf{V}}^p = \mathbf{R}\mathbf{U}^p\mathbf{R}^T \tag{5.8}$$

and

$$\bar{\mathbf{U}}^e = \mathbf{R}^T\mathbf{V}^e\mathbf{R} \tag{5.9}$$

we can write (5.7) in the alternative form

$$\mathbf{R} = \mathbf{R}^e\mathbf{R}^p \tag{5.10}$$

$$\mathbf{V} = \mathbf{V}^e\bar{\mathbf{V}}^p \tag{5.11}$$

$$\mathbf{U} = \bar{\mathbf{U}}^e\mathbf{U}^p. \tag{5.12}$$

Using (5.5), (5.8) and (5.10)

$$\bar{\mathbf{V}}^p = \mathbf{R}^e\mathbf{V}^p\mathbf{R}^{eT} \tag{5.13}$$

similarly using (5.6), (5.9) and (5.10)

$$\bar{\mathbf{U}}^e = \mathbf{R}^{pT}\mathbf{U}^e\mathbf{R}^p. \tag{5.14}$$

From the relations (5.8), (5.9), (5.13) and (5.14) it is easily shown that $\mathbf{V}^p$, $\mathbf{U}^p$ and $\bar{\mathbf{V}}^p$ all have the same principal values and if the principal vectors are v^p, u^p and $\bar{v}^p$ then they are related by

$$v^p = \mathbf{R}^p u^p \quad \text{and} \quad \bar{v}^p = \mathbf{R}u^p. \tag{5.15}$$

Similarly $\mathbf{V}^e$, $\mathbf{U}^e$ and $\bar{\mathbf{U}}^e$ all have the same principal values and if the principal vectors are v^e, u^e and $\bar{u}^e$ they are related by

$$v^e = \mathbf{R}^e u^e \quad \text{and} \quad \bar{u}^e = \boldsymbol{R}^T v^e = \boldsymbol{R}^{pT} u^e. \tag{5.16}$$

Applying the principle of material frame indifference to the energy it is clear that each energy term must depend only on objective tensors that are invariant under observer transformations and such a set of tensors are $\mathbf{U}$, $\bar{\mathbf{U}}^e$ and $\mathbf{U}^p$. Thus we will write (5.3) in the form

$$W = W^e(\bar{\mathbf{U}}^e) + W^p(\mathbf{U}^p) \tag{5.17}$$

$$= W^e(\mathbf{R}^{pT}\mathbf{U}^e\mathbf{R}^p) + W^p(\mathbf{U}^p). \tag{5.17a}$$

At first sight it seems strange that W^e must depend on $\bar{\mathbf{U}}^e$, and hence on $\mathbf{R}^p$, rather than on $\mathbf{U}^e$ alone. However, if the elastic response is anisotropic its elastic energy must contain orientation terms which will, under the above assumptions, be measured relative to the plastic configuration rather than the initial state.

Using equation (5.12) gives

$$W = W^e(\mathbf{U}\mathbf{U}^{p-1}) + W^p(\mathbf{U}^p) = W(\mathbf{U}, \mathbf{U}^p) \tag{5.18}$$

and then applying the third postulate gives

$$\left(\frac{\partial W}{\partial \mathbf{U}^p}\right)_{\mathbf{U}} = 0 \tag{5.19}$$

from which $\mathbf{U}^p$ can be found as a function of $\mathbf{U}$.

In the absence of thermal effects the energy equation takes the form

$$\mathbf{T}\dot{\mathbf{F}} = \rho\mathbf{F}\dot{W} \quad \text{or} \quad T_j^i\dot{F}_\alpha^j = \rho F_\alpha{}^i\dot{W} \tag{5.20}$$

where the dot denotes the material time derivative, $\mathbf{T}$ is the stress tensor and it is to be noted that W is the energy per unit mass.

Differentiating eqn (5.4) and substituting in (5.20) then gives

$$\mathbf{T}\dot{\mathbf{R}}\mathbf{U} + \mathbf{T}\mathbf{R}\dot{\mathbf{U}} = \rho\boldsymbol{R}\boldsymbol{U}\frac{\partial W}{\partial \mathbf{U}}\dot{\mathbf{U}}. \tag{5.21}$$

As **R** and **U** can be chosen independently $\dot{\mathbf{R}}$ must be zero and

$$\mathbf{R}^T\mathbf{T}\mathbf{R}\dot{\mathbf{U}} = \rho\mathbf{U}\frac{\partial W}{\partial \mathbf{U}}\dot{\mathbf{U}} \tag{5.22}$$

or

$$R_i^{T\alpha}T_j^iR_\beta^j\dot{U}_\gamma^\beta = \rho U_\gamma^\alpha\,\frac{\partial W}{\partial U_\nu^\mu}\,\dot{U}_\nu^\mu. \tag{5.23}$$

Letting

$$\tilde{\mathbf{T}} = \mathbf{R}^T\mathbf{T}\mathbf{R}/\rho$$

or

$$\tilde{T}_\beta^\alpha = R_i^{T\alpha}T_j^iR_\beta^j/\rho$$

gives

$$\tilde{\mathbf{T}} = \mathbf{U}\frac{\partial W}{\partial \mathbf{U}} \quad \text{or} \quad \tilde{T}_\beta^\alpha = U_\gamma^\alpha\,\frac{\partial W}{\partial U_\gamma^\beta}. \tag{5.24}$$

Using (5.18) and (5.19) we can write

$$W = W(\mathbf{U}, \mathbf{U}^p(\mathbf{U}))$$

so that

$$\frac{\partial W}{\partial \mathbf{U}} = \left(\frac{\partial W}{\partial \mathbf{U}}\right)_{\mathbf{U}^p} + \left(\frac{\partial W}{\partial \mathbf{U}^p}\right)_{\mathbf{U}}\frac{\partial \mathbf{U}^p}{\partial \mathbf{U}} \tag{5.25}$$

then using (5.19)

$$\frac{\partial W}{\partial \mathbf{U}} = \left(\frac{\partial W}{\partial \mathbf{U}}\right)_{\mathbf{U}^p}. \tag{5.26}$$

But using eqn (5.12) with $\mathbf{U}^p$ regarded as constant gives

$$\left(\frac{\partial}{\partial \mathbf{U}}\right)_{\mathbf{U}^p} = (\mathbf{U}^p)^{-1}\,\frac{\partial}{\partial \bar{\mathbf{U}}^e} \tag{5.27}$$

From (5.24) (5.26) and (5.27)

$$\tilde{\mathbf{T}} = \mathbf{U}\frac{\partial W}{\partial \mathbf{U}} = \mathbf{U}(\mathbf{U}^p)^{-1}\frac{\partial W}{\partial \bar{\mathbf{U}}^e} = \bar{\mathbf{U}}^e\frac{\partial W}{\partial \bar{\mathbf{U}}^e} = \bar{\mathbf{U}}^e\frac{\partial W^e}{\partial \bar{\mathbf{U}}^e}. \tag{5.28}$$

Thus, as we have found previously, the stress depends only on the elastic strain energy. From the definitions of $\tilde{\mathbf{T}}$ and $\bar{\mathbf{U}}^e$ eqn (5.28) can also be written in the form

$$\mathbf{T} = \rho\mathbf{V}^e\frac{\partial W^e}{\partial \mathbf{V}^e} \quad \text{or} \quad T_j^i = \rho V_k^{ei}\,\frac{\partial W^e}{\partial V_k^{ej}}. \tag{5.29}$$

It is shown in texts on continuum mechanics that

$$\det \mathbf{U} = \rho_0/\rho = \det \bar{\mathbf{U}}^e \det \mathbf{U}^p \tag{5.30}$$

where ρ_0 is the initial density. Incompressibility is therefore expressed as $\det \mathbf{U} = 1$ rather than the infinitesimal condition $Tr\epsilon = 0$. It thus seems expedient to replace the division of ϵ into an isotropic and deviatoric part by the following decomposition.

$$\mathbf{U} = U^{xx}\mathbf{U}^{x}; \qquad \bar{\mathbf{U}}^{e} = \bar{U}^{exx}\bar{\mathbf{U}}^{ex}; \qquad \mathbf{U}^{p} = U^{pxx}\mathbf{U}^{px} \tag{5.31a–c}$$

where

$$\det \mathbf{U}^{x} = 1; \qquad \det \bar{\mathbf{U}}^{ex} = 1; \qquad \det \mathbf{U}^{px} = 1 \tag{5.32a–c}$$

substituting (5.31) into (5.12), taking determinants and using (5.32) gives

$$U^{xx} = \bar{U}^{exx}U^{pxx} \tag{5.33}$$

then using (5.12), (5.13) and (5.33)

$$\mathbf{U}^{x} = \bar{\mathbf{U}}^{ex}\mathbf{U}^{px} \tag{5.34}$$

taking the determinant of both sides of (5.31a) gives

$$U^{xx} = (\det \mathbf{U})^{1/3} \tag{5.35}$$

$$\mathbf{U}^{x} = \mathbf{U}(\det \mathbf{U})^{-1/3}. \tag{5.36}$$

Now let us find the effect of this decomposition on the stress. From eqn (5.24)

$$\bar{\mathbf{T}} = \mathbf{U}\frac{\partial W}{\partial \mathbf{U}} \quad \text{or} \quad \bar{T}_{\beta}{}^{\alpha} = U_{\gamma}{}^{\alpha}\frac{\partial W}{\partial U_{\gamma}{}^{\beta}} \tag{5.37}$$

Now

$$U_{\gamma}{}^{\alpha}\frac{\partial}{\partial U_{\gamma}{}^{\beta}} = U_{\gamma}{}^{\alpha}\frac{\partial U^{xx}}{\partial U_{\gamma}{}^{p}}\frac{\partial}{\partial U^{xx}} + U_{\gamma}{}^{\alpha}\frac{\partial U_{\nu}{}^{x\mu}}{\partial U_{\gamma}{}^{\beta}}\frac{\partial}{\partial U_{\nu}{}^{x\mu}}$$

and

$$\mathrm{U}\gamma^{\alpha}\frac{\partial}{\partial U_{\gamma}{}^{\beta}}\det \mathbf{U} = U_{\gamma}{}^{\alpha}\mathcal{U}_{\beta}{}^{\gamma}$$

where $\mathcal{U}_{\beta}{}^{\gamma}$ is the co-factor of $U_{\gamma}{}^{\beta}$ in the determinant. By an elementary theorem in algebra this product is zero unless α equals β and then the product is $\det \mathbf{U}$.
Thus

$$U_{\gamma}{}^{\alpha}\frac{\partial \det \mathbf{U}}{\partial U_{\gamma}{}^{\beta}} = \delta_{\alpha\beta}\det \mathbf{U}. \tag{5.38}$$

Therefore using (5.35)

$$U_{\gamma}{}^{\alpha}\frac{\partial U^{xx}}{\partial U_{\gamma}{}^{\beta}}\frac{\partial}{\partial U^{xx}} = \frac{1}{3}\delta_{\alpha\beta}U^{xx}\frac{\partial}{\partial U^{xx}}. \tag{5.39}$$

Also

$$\begin{aligned} U_{\gamma}{}^{\alpha}\frac{\partial U_{\nu}{}^{x\mu}}{\partial U_{\gamma}{}^{\beta}}\frac{\partial}{\partial U_{\nu}{}^{x\mu}} &= U_{\gamma}{}^{\alpha}\left[-\frac{1}{U^{xx2}}\frac{\partial U^{xx}}{\partial U_{\gamma}{}^{\beta}}U_{\nu}{}^{\mu} + \frac{1}{U^{xx}}\frac{\partial U_{\nu}{}^{\mu}}{\partial U_{\gamma}{}^{\beta}}\right]\frac{\partial}{\partial U_{\nu}{}^{x\mu}} \\ &= -\frac{1}{3}\delta_{\alpha\beta}U_{\nu}{}^{x\mu}\frac{\partial}{\partial U_{\nu}{}^{x\mu}} + U_{\gamma}{}^{x\alpha}\frac{\partial}{\partial U_{\gamma}{}^{x\beta}}. \end{aligned} \tag{5.40}$$

Using eqns (5.37)–(5.40)

$$\tilde{T}_\beta{}^\alpha = \frac{1}{3} U^{xx} \frac{\partial W}{\partial U^{xx}} \delta_{\alpha\beta} + \left\{ U_\gamma{}^{x\alpha} \frac{\partial W}{\partial U_\gamma{}^{x\beta}} - \frac{1}{3} U_\nu{}^{x\mu} \frac{\partial W}{\partial U_\nu{}^{x\mu}} \delta_{\alpha\beta} \right\}. \tag{5.41}$$

The trace of the second term on the right hand side of this equation vanishes and thus the proposed decomposition of **U** implies a decomposition of $\tilde{\mathbf{T}}$ into isotropic and deviatoric parts. Using our previous notation

$$\tilde{T}'' = \frac{1}{3} U^{xx} \frac{\partial W}{\partial U^{xx}} \tag{5.42}$$

$$\tilde{T}'_\beta{}^\alpha = \left\{ U_\gamma{}^{x\alpha} \frac{\partial W}{\partial U_\gamma{}^{x\beta}} - \frac{1}{3} U_\nu{}^{x\mu} \frac{\partial W}{\partial U_\nu{}^{x\mu}} \delta_{\alpha\beta} \right\}. \tag{5.43}$$

Using eqn (5.28) and following the same argument as above we can show that eqns (5.42) and (5.43) apply if we replace W by W^e and **U** by $\bar{\mathbf{U}}^e$ giving

$$\tilde{T}'' = \frac{1}{3} \bar{U}^{exx} \frac{\partial W^e}{\partial \bar{U}^{exx}} \tag{5.44}$$

$$\tilde{T}'_\beta{}^\alpha = \left\{ \bar{U}_\gamma{}^{ex\alpha} \frac{\partial W^e}{\partial \bar{U}_\gamma{}^{ex\beta}} - \frac{1}{3} \bar{U}_\nu{}^{ex\mu} \frac{\partial W^e}{\partial \bar{U}_\nu{}^{ex\mu}} \delta_{\alpha\beta} \right\}. \tag{5.45}$$

To find the minimum value of the total energy density

$$W = W^e(\bar{\mathbf{U}}^e) + W^p(\mathbf{U}^p). \tag{5.46}$$

We set each partial derivative $\mathbf{U}^p$ to zero

$$\left(\frac{\partial W}{\partial U_\beta{}^{p\alpha}} \right)_{\mathrm{U}} = \frac{\partial W^e}{\partial \bar{U}_\gamma{}^{e\mu}} \left(\frac{\partial \bar{U}_\nu{}^{e\mu}}{\partial U_\beta{}^{p\alpha}} \right) + \frac{\partial W^p}{\partial U_\beta{}^{p\alpha}} = 0. \tag{5.47}$$

Now from eqn (5.12)

$$\left(\frac{\partial \bar{U}_\nu{}^{e\mu}}{\partial U_\beta{}^{p\alpha}} \right)_U = - \bar{U}_\alpha{}^{e\mu} (U^p)_\nu{}^{-1\beta} \tag{5.48}$$

thus

$$\left(\frac{\partial \bar{U}_\nu{}^{e\mu}}{\partial U_\beta{}^{p\alpha}} \right)_{\mathrm{U}} U_\beta{}^{p\alpha} = - \bar{U}_\nu{}^{e\mu}. \tag{5.49}$$

Multiplying (5.47) by $U_\beta{}^{p\alpha}$ then gives

$$\bar{U}_\nu{}^{e\mu} \frac{\partial W^e}{\partial \bar{U}_\nu{}^{e\mu}} = U_\beta{}^{p\alpha} \frac{\partial W^p}{\partial U_\beta{}^{p\alpha}} \left[= \tilde{\mathbf{T}} \quad \text{by eqn (5.28)} \right] \tag{5.50}$$

as the minimum energy condition.

As we have no guide as to what form W^p must take in general we will from henceforth restrict our discussion to isotropic materials. In this case W^e and W^p are functions of the principal invariants of $\bar{\mathbf{U}}^e$ and $\mathbf{U}^p$ respectively.
Thus

$$\frac{\partial W^e}{\partial \bar{U}_\nu{}^{e\mu}} = \frac{\partial W^e}{\partial I_e} \frac{\partial I_e}{\partial \bar{U}_\nu{}^{e\mu}} + \frac{\partial W^e}{\partial II_e} \frac{\partial II_e}{\partial \bar{U}_\nu{}^{e\mu}} + \frac{\partial W^e}{\partial III_e} \frac{\partial III_e}{\partial \bar{U}_\nu{}^{e\mu}} \tag{5.51}$$

and

$$\frac{\partial W^p}{\partial U_\beta{}^{p\alpha}} = \frac{\partial W^p}{\partial I_p}\frac{\partial I_p}{\partial U_\beta{}^{p\alpha}} + \frac{\partial W^p}{\partial II_p}\frac{\partial II_p}{\partial U_\beta{}^{p\alpha}} + \frac{\partial W^p}{\partial III_p}\frac{\partial III_p}{\partial U_\beta{}^{p\alpha}}. \tag{5.52}$$

Substituting in (5.50) and using Eulers theorem on homogenous functions

$$\frac{\partial W^e}{\partial I_e} I_e + 2\frac{\partial W^e}{\partial II_e} II_e + 3\frac{\partial W^e}{\partial III_e} III_e = \frac{\partial W^p}{\partial I_p} I_p + 2\frac{\partial W^p}{\partial II_p} II_p + 3\frac{\partial W^p}{\partial III_p} III_p. \tag{5.53}$$

If we make the usual assumption that plastic deformation occurs incompressibly then U^{pxx} is unity and $\bar{U}^{exx}$ equals U^{xx}. For a given U^{xx}, $\bar{U}^{exx}$ is thus fixed and from eqn (5.42) it is clear that the isotropic or hydrostatic stress will have no effect on the yield stress.

As the state of minimum energy in the absence of a disturbing force corresponds to $\bar{\mathbf{U}}^e = \mathbf{U}^p = \mathbf{1}$ it is convenient to define new strain tensors as follows

$$\bar{\boldsymbol{\varepsilon}}^e = \bar{\mathbf{U}}^e - \mathbf{1} \tag{5.54}$$

$$\boldsymbol{\varepsilon}^p = \mathbf{U}^p - \mathbf{1} \tag{5.55}$$

$$\boldsymbol{\varepsilon} = \mathbf{U} - \mathbf{1}. \tag{5.56}$$

If $\boldsymbol{\varepsilon}$ is very small then we can let

$$\varepsilon^{xx} = U^{xx} - 1 \tag{5.57}$$

and

$$\boldsymbol{\varepsilon}^x = \mathbf{U}^x - \mathbf{1}. \tag{5.58}$$

Using (5.31) and neglecting second order small quantities

$$\boldsymbol{\varepsilon} = \varepsilon^{xx}\mathbf{1} + \boldsymbol{\varepsilon}^x. \tag{5.59}$$

Thus for small deformations $\varepsilon^{xx} \to \varepsilon''$ and $\boldsymbol{\varepsilon}^x \to \boldsymbol{\varepsilon}'$ and to the same order of accuracy

$$\tilde{T}'' = \frac{1}{3}\frac{\partial W}{\partial \epsilon''} = \frac{1}{3}\frac{\partial W^e}{\partial \bar{\epsilon}^{e''}} \tag{5.60}$$

and

$$\tilde{\mathbf{T}}' = \frac{\partial W}{\partial \boldsymbol{\epsilon}'} = \frac{\partial W^e}{\partial \bar{\boldsymbol{\epsilon}}^{e'}} \tag{5.61}$$

which corresponds directly with eqn (4.10).

To follow the analogy with previous sections further we would require simple expressions for the elastic and plastic energy components. However, relatively little has been done in this direction particularly for the plastic energy term. We will therefore set up a system which has an obvious analogy with the previous section. For this it is convenient to re-write eqn (5.53) in a different form. If we decompose $\boldsymbol{\varepsilon}$ into an istropic and deviatoric part we can write (5.56) in the form

$$\mathbf{U} = (1 + \varepsilon'')\mathbf{1} + \boldsymbol{\varepsilon}'. \tag{5.62}$$

Then after some algebra it can then be shown

$$I\frac{\partial}{\partial I} + 2II\frac{\partial}{\partial II} + 3III\frac{\partial}{\partial III} = (1 + \varepsilon'')\frac{\partial}{\partial \varepsilon''} + 2II'\frac{\partial}{\partial II'} + 3III'\frac{\partial}{\partial III'} \tag{5.63}$$

where I, II, III are the first, second and third invariants of $\mathbf{U}$ and II', III' are the second and third invariants of $\boldsymbol{\varepsilon}'$. Thus finally eqn (5.53) can be written in the form

$$(1+\bar{\varepsilon}^{e*})\frac{\partial W^e}{\partial \bar{\varepsilon}^{e*}} + 2II'_e\frac{\partial W^e}{\partial II'_e} + 3III'_e\frac{\partial W^e}{\partial III'_e} = (1+\varepsilon^{p*})\frac{\partial W^p}{\partial \varepsilon^{p*}} + 2II'_p\frac{\partial W^p}{\partial II'_p} + 3III'_p\frac{\partial W^p}{\partial III'_p} \qquad (5.64)$$

which promises to be a useful form of the criterion for determining the yield surface when a suitable expression for the plastic energy density becomes available.

6. CONCLUDING REMARKS

The theory of plasticity proposed in this paper is based on the three postulates enunciated in Section 2, whose physical motivation is also given in that section. These postulates provide a useful adjunct to the general principles of continuum mechanics and, as we have shown, the theory appears to have the potential to model most of the main features of plasticity in situations where dynamic and thermal effects are not dominant. The sharp changes in the character of the stress–strain relation is reminiscent of the catastrophes of Thom[20] and this aspect of the theory may be worth exploring further.

REFERENCES

1. R. Hill, *The Mathematical Theory of Plasticity*. Clarendon Press, Oxford. (1950).
2. E. R. Fitzgerald, *Particle Waves and Deformation in Crystalline Solids*. Wiley-Interscience, New York (1966).
3. G. Wempner and J. Aberson, *Int. J. Solids Structures* **12**, 705–721 (1976).
4. J. F. Bell, *The Physics of Large Deformation of Crystalline Solids*. Wiley, New York (1968).
5. J. F. Bell, The experimental foundations of solid mechanics, *Mechanics of Solids* 1. (Edited by C. Truesdell). *Handbuch der Physik* Vol. VIa/1 (1973).
6. A. Phillips and G. A. Gray, *Trans. ASME Ser. D. J. Basic Eng.* **83**, 275–288 (1961).
7. J. J. Gilman, *Micromechanics of Flow in Solids*. McGraw-Hill, New York (1969).
8. B. Budiansky, *J. Appl. Mech.* **26**. 259–264 (1950).
9. C. Truesdell and R. Toupin, The classical field theories. *Handbuch der Physik III/1* Springer, Berlin (1960).
10. C. Truesdell and W. Noll, The non-linear field theories of mechanics. *Handbuch der Physik III/3*. Springer, Berlin (1965).
11. A. E. Green and P. M. Naghdi, *Arch. Rational Mech. Anal.* **18**, 251–281 (1965).
12. E. H. Lee, *J. Appl. Mech* **36**, 1–6 (1969).
13. R. Hill, *Proc. R. Soc. Lond.* **A326**, 131–147 (1972).
14. T. W. Ting, Topics in the Mathematical Theory of Plasticity, *Handbuch der Physik* Vol. VIa/3. (Edited by C. Truesdell). Springer, Berlin.
15. H. Geiringer, Ideal plasticity. (Edited by C. Truesdell). *Handbuch der Physik* Vol. VIa/3. *Mechanics of Solids* 3 (1973).
16. F. R. N. Nabarro, *Theory of Crystal Dislocations*. Clarendon Press, Oxford (1967).
17. P. Glansdorf and I. Prigogine, *Thermodynamic Theory of Structure, Stability and Fluctuations*. Wiley-Interscience, New York (1971).
18. E. W. Billington and A. Tate, *Proc. R. Soc. Lond.* **A327**, 23–46 (1972).
19. T. H. Lin and M. Ito, *J. Mech. Phys. Solids* **13**, 103–115 (1965).
20. R. Thom, *Structural Stability and Morphogensis*. Benjamin, Reading, Massachussetts (1975).